On Certain Unsteady Torsional Flows of Simple Fluids

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This paper presents an investigation of the equations of motion to find the conditions under which these equations admit exact viscometric solutions for certain unsteady torsional flows of incompressible simple fluids. In general, for the cone and plane and the parallel disk geometries, curvilineal solutions do not exist for *arbitrary* members of the class of simple fluids; however, if the inertia terms can be neglected in the equations of motion, curvilineal solutions can be found for the complete class of simple fluids.

The use of torsional viscometers for the measurement of fluid properties in small amplitude, small frequency periodic flows suggests that the theoretical basis for these experiments be established in general terms. This requires that the equations of motion be investigated to find the conditions under which exact solutions of these equations exist. The results of such an investigation are presented in this paper for certain unsteady torsional flows of incompressible simple fluids.

The constitutive equation for an incompressible simple fluid and its reduced form for special classes of flows are given in a treatise by Truesdell and Noll (2). Their discussion of the presently known exact viscometric solutions of the equations of motion suggests the method of analysis used here. Some results from the theory of curvilineal flows are summarized, and then these are applied to unsteady torsional flows in the cone and plane and the parallel disk geometries.

CURVILINEAL FLOWS

A flow is a member of the class of curvilineal flows if its velocity field has a representation

$$v^1 = 0 \tag{1a}$$

$$v^2 = 0 \tag{1b}$$

$$v^3 = \omega(x^2, t) \tag{1c}$$

in an orthogonal coordinate system x^1, x^2, x^3 for which the lengths of the base vectors are constant along the x^3 coordinate curves; that is for which

$$h_i = h_i(x^1, x^2) \quad i = 1, 2, 3 \tag{2}$$

For all such flows, the deviatoric stress field in an incompressible simple fluid is prescribed by the three scalar valued functionals $\mathfrak{T}$, $\mathfrak{S}_1$, and $\mathfrak{S}_2$, which completely determine the physical components of the extra stress tensor in the following way:

$$S<12> = 0 \tag{3a}$$

$$S<13> = 0 \tag{3b}$$

$$S<23> = \mathop{\mathfrak{T}}_{s=-\infty}^{s=0}\Big(\lambda(s)\Big) \tag{3c}$$

$$S<22> - S<11> = \mathop{\mathfrak{S}_1}_{s=-\infty}^{s=0}\Big(\lambda(s)\Big) \tag{3d}$$

$$S<33> - S<11> = \mathop{\mathfrak{S}_2}_{s=-\infty}^{s=0}\Big(\lambda(s)\Big) \tag{3e}$$

where

$$\lambda(s) = \int_0^s \frac{h_3(x^1, x^2)}{h_2(x^1, x^2)} \frac{\partial}{\partial x^2}\, \omega(x^2, t+\tau)\,d\tau \tag{4}$$

By use of Equations (1), (2), and (3), the covariant components of the equation of motion can be written in the following form:

$$-\rho h_3 \frac{\partial h_3}{\partial x^1} v^3 v^3 = -\frac{\partial p}{\partial x^1} + \frac{\partial S<11>}{\partial x^1} + \frac{1}{h_2}\frac{\partial h_2}{\partial x^1}(S<11> - S<22>) + \frac{1}{h_3}\frac{\partial h_3}{\partial x^1}(S<11> - S<33>) \tag{5a}$$

$$-\rho h_3 \frac{\partial h_3}{\partial x^2} v^3 v^3 = -\frac{\partial p}{\partial x^2} + \frac{\partial S<22>}{\partial x^2} + \frac{1}{h_1}\frac{\partial h_1}{\partial x^2}(S<22> - S<11>) + \frac{1}{h_3}\frac{\partial h_3}{\partial x^2}(S<22> - S<33>) \tag{5b}$$

$$\rho h_3^2 \frac{\partial v^3}{\partial t} = -\frac{\partial p}{\partial x^3} + \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x^2}(h_1 h_3^2\, S<32>) \tag{5c}$$

where the body force vector has been taken to be 0. From Equations (5a), (5b), and (5c)

$$\frac{\partial^2 p}{\partial x^3 \partial x^1} = \frac{\partial^2 p}{\partial x^3 \partial x^2} = \frac{\partial^2 p}{\partial x^3 \partial x^3} = 0$$

or

$$\frac{\partial^2 p}{\partial x^1 \partial x^3} = \frac{\partial^2 p}{\partial x^2 \partial x^3} = \frac{\partial^2 p}{\partial x^3 \partial x^3} = 0$$

and therefore $p = x^3 f(t) + g(x^1, x^2, t)$. For the particular flows to be investigated here, p satisfies the periodicity condition $p(x^1, x^2, x^3, t) = p(x^1, x^2, x^3 + 2\pi, t)$, and thus $f(t) = 0$ or $(\partial p)/(\partial x^3) = 0$.

CONE AND PLANE GEOMETRY

We investigate first the conditions under which an unsteady torsional flow of the form

$$v^r = 0,\ v^\theta = 0,\ v^\phi = \omega(\theta, t) \tag{6}$$

can be an exact solution of Equations (5a), (5b), and (5c), where x^1, x^2, and x^3 are taken to be the spherical coordinates r, θ, ϕ, and $h_1 = 1$, $h_2 = r$, $h_3 = r \sin\theta$. This flow is easily seen to be a curvilineal flow, and we may therefore proceed directly to examine the equations of motion.

From Equations (3) and (4) we find that

$$\lambda(s) = \int_0^s \sin\theta \frac{\partial \omega}{\partial \theta}(\theta, t+\tau)\,d\tau$$

and that the physical components of $\tilde{S}$ depend only on θ, t. As a result Equations (5a), (5b), and (5c) become

$$-\rho r \sin^2\theta\, \omega(\theta, t)^2 = -\frac{\partial p}{\partial r} + \frac{2S<rr> - S<\theta\theta> - S<\phi\phi>}{r} \tag{8a}$$

$$-\rho r^2 \sin\theta \cos\theta\, \omega(\theta, t)^2 = -\frac{\partial p}{\partial \theta} + \frac{\partial S<\theta\theta>}{\partial \theta} + \frac{\cos\theta}{\sin\theta}(S<\theta\theta> - S<\phi\phi>) \tag{8b}$$

$$\rho r^2 \sin^2\theta \frac{\partial \omega(\theta, t)}{\partial t} = \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin^2\theta\, S<\theta\phi> \tag{8c}$$

and we see that an exact solution can exist only if there are two functions $\omega(\theta, t)$ and $p(r, \theta, t)$ which identically satisfy Equations (8a), (8b), and (8c).

Equations (3c), (7), and (8c) can be combined to give the following differential functional equation for $\omega(\theta, t)$:

$$\rho r^2 \sin^2\theta \frac{\partial \omega(\theta, t)}{\partial t} = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}$$

$$\left[\sin^2\theta \mathop{\mathfrak{T}}_{s=-\infty}^{s=0} \left(\int_0^s \sin\theta \frac{\partial \omega}{\partial \theta} (\theta, t+\tau) d\tau \right) \right] \tag{9}$$

which clearly cannot be satisfied exactly by any function $\omega(\theta, t)$ when $\rho > 0$. As a result none of the flows defined by Equation (6) can be an exact solution of the equations of motion unless $\rho = 0$.

In this latter case, or for negligible inertia, Equation (9) can be integrated to

$$S\langle\theta\phi\rangle = \mathop{\mathfrak{T}}_{s=-\infty}^{s=0} \left(\int_0^s \sin\theta \frac{\partial \omega}{\partial \theta} (\theta, t+\tau) d\tau \right) = \frac{f(t)}{\sin^2\theta} \tag{10}$$

and this equation together with the boundary conditions

$$\omega\left(\frac{1}{2}\pi - \theta_o, t\right) = \Omega(t) \tag{11a}$$

$$\omega\left(\frac{1}{2}\pi, t\right) = 0 \tag{11b}$$

can be expected to determine infinitely many solutions. For any of these solutions the physical components of $\underset{\sim}{S}$ can be found, and thus a flow of the class defined by Equation (6) can be shown to exist if a function $p(r, \theta, t)$ satisfying Equations (8a) and (8b) can be shown to exist. Such a function exists if the integrability condition $(\partial^2 p)/(\partial r \partial \theta) = (\partial^2 p)/(\partial \theta \partial r)$ is satisfied, and this condition is equivalent to

$$\frac{\partial}{\partial \theta} (2S\langle rr\rangle - S\langle\theta\theta\rangle - S\langle\phi\phi\rangle)$$

$$= \frac{\partial}{\partial \theta} \left[\mathop{\mathfrak{S}_1}_{s=-\infty}^{s=0} \left(\lambda(s) \right) + \mathop{\mathfrak{S}_2}_{s=-\infty}^{s=0} \left(\lambda(s) \right) \right] = 0 \tag{12}$$

Equation (12) limits the exact solutions for the case $\rho = 0$ to a very special class of simple fluids; however, this condition can be removed if we make the approximation that $\theta_o \approx 0$ or that $\sin\theta \approx 1$, $\pi/2 - \theta_o < \theta < \pi/2$. In this approximation, Equation (10) becomes

$$\mathop{\mathfrak{T}}_{s=-\infty}^{s=0} \left[\int_0^s \frac{\partial \omega}{\partial \theta} (\theta, t+\tau) d\tau \right] = f(t)$$

and a particular solution to this equation and Equation (11) is

$$\omega(\theta, t) = \frac{1/2\,\pi - \theta}{\theta_o} \Omega(t) \tag{13a}$$

where

$$\lambda(s) = \int_0^s -\frac{1}{\theta_o} \Omega(t+\tau) d\tau$$

and

$$S\langle\theta\phi\rangle = f(t) = \mathop{\mathfrak{T}}_{s=-\infty}^{s=0} \left[\int_0^s -\frac{1}{\theta_o} \Omega(t+\tau) d\tau \right] \tag{13b}$$

The physical components of $\underset{\sim}{S}$ depend only on t, and therefore the compatibility condition, Equation (12), reduces to $0 = 0$ for all simple fluids. Equation (8a) reduces in this limit to

$$\frac{\partial T\langle\theta\theta\rangle}{\partial \ln r} = \mathop{\mathfrak{S}_1}_{s=-\infty}^{s=0} \left(\lambda(s) \right) + \mathop{\mathfrak{S}_2}_{s=-\infty}^{s=0} \left(\lambda(s) \right) \tag{13c}$$

which is a convenient equation for the interpretation of experimental data.

The particular solution given here includes as a special case the steady flow solution given by Truesdell and Noll (2); that is, if Ω = const and $\omega = \omega(\theta)$, then

$$\lambda(s) = \int_0^s \sin\theta\, \omega'(\theta) d\tau = \sin\theta\, \omega'(\theta)_s = \kappa s$$

$$\mathop{\mathfrak{T}}_{s=-\infty}^{s=0} \left(\lambda(s) \right) = \tau(\kappa)$$

$$\mathop{\mathfrak{S}_i}_{s=-\infty}^{s=0} \left(\lambda(s) \right) = \sigma_i(\kappa) \quad i = 1, 2$$

and for $\theta_o \approx 0$

$$\omega(\theta) = \frac{\frac{\pi}{2} - \theta}{\theta_o} \Omega$$

and

$$S\langle\theta\phi\rangle = \tau\left(-\frac{\Omega}{\theta_o}\right)$$

The unsteady solutions are therefore physically reasonable at least under conditions such that the fluid memories are so short or the variation of Ω so slow that $\Omega(t + \tau)$ can be replaced by $\Omega(t)$ for $-\infty < \tau < 0$. It should be noted that the particular solution given by Equations (13a), (13b), and (13c) is not completely independent of whatever conditions may have been imposed at earlier times. The velocity field however is not only independent of the initial conditions but also of the mechanical properties of the material, just as it is in steady simple shearing motions.

We investigate briefly the conditions under which an unsteady torsional solenoidal flow of the form

$$v^r = 0, \ v^\theta = 0, \ v^\phi = \omega(r, \theta, t) \tag{14}$$

can be an exact solution of Equations (5a), (5b), and (5c). This flow, investigated by Williams and Bird (3) and Nally (1), removes the difficulties presented by Equation (9).

The flow defined by Equation (14) is a curvilineal flow, but not in spherical coordinates. If we suppose that the surfaces of constant angular velocity at time t' coincide with those at time t, then we can introduce a coordinate system $\bar{x}^1, \bar{x}^2, \bar{x}^3$ defined by

$$\bar{x}^1 = \xi(r, \theta, t')$$

$$\bar{x}^2 = \omega(r, \theta, t')$$

$$\bar{x}^3 = \phi$$

where t' is an arbitrary fixed time and $\xi(r, \theta, t')$ satisfies the condition

$$\frac{\partial \xi}{\partial r} \frac{\partial \omega}{\partial r} + \frac{1}{r^2} \frac{\partial \xi}{\partial \theta} \frac{\partial \omega}{\partial \theta} = 0$$

which is the necessary and sufficient condition that the three families of surfaces x^k = const be mutually orthogonal.

In the $\bar{x}^1, \bar{x}^2, \bar{x}^3$ coordinate system

$$\bar{v}^1 = 0$$
$$\bar{v}^2 = 0$$
$$\bar{v}^3 = \bar{v}^3 = \bar{\omega}(\bar{x}^2, t)$$
$$\bar{h}_i = \bar{h}_i(\bar{x}^1, \bar{x}^2) \quad i = 1, 2, 3$$

and

$$\cos\alpha = \frac{\frac{1}{r}\frac{\partial\omega}{\partial\theta}(r, \theta, t')}{\sqrt{\left(\frac{\partial\omega}{\partial r}(r, \theta, t')\right)^2 + \left(\frac{1}{r}\frac{\partial\omega}{\partial\theta}(r, \theta, t')\right)^2}}$$

where α is the angle between the r and $\bar{x}^1$ coordinate lines and is independent of the choice of t'. The flow is easily seen to be a curvilineal flow in the $\bar{x}^1, \bar{x}^2, \bar{x}^3$ coordinate system, and therefore the physical components of the extra stress are given by Equation (3). If we express these results in the r, θ, ϕ coordinate system, we have

$$S<r\theta> = \sin\alpha \cos\alpha \mathop{\mathfrak{S}_1}_{s=-\infty}^{s=0}\left(\lambda(s)\right)$$

$$S<r\phi> = \sin\alpha \mathop{\mathfrak{T}}_{s=-\infty}^{s=0}\left(\lambda(s)\right)$$

$$S<\theta\phi> = \cos\alpha \mathop{\mathfrak{T}}_{s=-\infty}^{s=0}\left(\lambda(s)\right)$$

$$S<\theta\theta> - S<rr> = \cos 2\alpha \mathop{\mathfrak{S}_1}_{s=-\infty}^{s=0}\left(\lambda(s)\right)$$

$$2S<\phi\phi> - S<rr> - S<\theta\theta>$$

$$= 2 \mathop{\mathfrak{S}_2}_{s=-\infty}^{s=0}\left(\lambda(s)\right) - \mathop{\mathfrak{S}_1}_{s=-\infty}^{s=0}\left(\lambda(s)\right)$$

where

$$\lambda(s) =$$
$$\int_0^s r\sin\theta\sqrt{\left(\frac{\partial\omega}{\partial r}(r,\theta,t+\tau)\right)^2 + \left(\frac{1}{r}\frac{\partial\omega}{\partial\theta}(r,\theta,t+\tau)\right)^2}\, d\tau$$

The ϕ component of the equation of motion

$$\rho r^2 \sin^2\theta \frac{\partial\omega(r,\theta,t)}{\partial t} = \frac{\sin\theta}{r^2}\frac{\partial}{\partial r}(r^2 S<r\phi>) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\, S<\theta\phi>) \quad (15)$$

may be reduced to a single differential functional equation for $\omega(r, \theta, t)$, and this, unlike Equation (9), may be expected to determine infinitely many solutions. However, for this flow the integrability condition $(\partial^2 p)/(\partial r\partial\theta) = (\partial^2 p)/(\partial\theta\partial r)$ for the r and θ components of the equation of motion leads to the following generalization of Equation (12):

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^3 S<r\theta>) + \frac{\partial S<\theta\theta>}{\partial\theta} + \frac{\cos\theta}{\sin\theta}(S<\theta\theta> - S<\phi\phi>) + \rho r^2 \sin\theta\cos\theta\, \omega(r,\theta,t)^2\right) - \frac{\partial}{\partial\theta}\left(\frac{\partial S<rr>}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, S<r\theta>) + \frac{2S<rr> - S<\theta\theta> - S<\phi\phi>}{r} + \rho r \sin^2\theta\, \omega(r,\theta,t)^2\right) = 0 \quad (16)$$

For special constitutive theories, Williams and Bird (3) and Nally (1) give solutions to Equation (15), but in neither case is Equation (16) satisfied exactly for $\rho > 0$.

PARALLEL DISK GEOMETRY

The parallel disk geometry is similar to the cone and plane geometry with $\theta_0 \approx 0$. We investigate the conditions under which an unsteady torsional flow of the form

$$v^r = 0, v^z = 0, v^\phi = \omega(z, t) \quad (17)$$

can be an exact solution of Equations (5*a*), (5*b*), and (5*c*), where now x^1, x^2, x^3 are taken to be the cylindrical coordinates r, z, ϕ, and $h_1 = 1$, $h_2 = 1$, $h_3 = r$. This flow is a curvilineal flow, and we may again proceed directly to examine the equations of motion. For this flow

$$\lambda(s) = \int_0^s r\frac{\partial\omega}{\partial z}(z, t+\tau)\, d\tau \quad (18)$$

The physical components of $\underset{\sim}{S}$ appear to depend on r, z, t, and Equations (5*a*), (5*b*), and (5*c*) reduce to

$$-\rho r\omega(z,t)^2 = -\frac{\partial p}{\partial r} + \frac{\partial S<rr>}{\partial r} + \frac{S<rr> - S<\phi\phi>}{r} \quad (19a)$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial S<zz>}{\partial z} \quad (19b)$$

$$\rho r^2\frac{\partial\omega(z,t)}{\partial t} = r\frac{\partial S<z\phi>}{\partial z} \quad (19c)$$

Equations (3*c*), (18), and (19*c*) can be combined to give

$$\rho r\frac{\partial\omega(z,t)}{\partial t} = \frac{\partial}{\partial z}\left[\mathop{\mathfrak{T}}_{s=-\infty}^{s=0}\left(\int_0^s r\frac{\partial\omega}{\partial z}(z, t+\tau)\,d\tau\right)\right]$$

and this equation together with the boundary conditions

$$\omega(0, t) = \Omega(t) \quad (20a)$$

$$\omega(b, t) = 0 \quad (20b)$$

can be expected to determine solutions $\omega(z, t)$ if the functional $\mathfrak{T}$, satisfies the condition

$$\mathop{\mathfrak{T}}_{s=-\infty}^{s=0}\left(a\lambda(s)\right) = a\mathop{\mathfrak{T}}_{s=-\infty}^{s=0}\left(\lambda(s)\right) \quad (21)$$

where a = const. If Equation (21) is satisfied, the integrability condition for Equations (19*a*) and (19*b*) is

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r}(S<rr> - S<zz>) + \frac{1}{r}(S<rr> - S<\phi\phi>) + \rho r\omega(z,t)^2\right) = 0 \quad (22)$$

and this reduces to

$$\frac{\partial}{\partial z}\left(2S<rr> - S<zz> - S<\phi\phi> + \rho r\omega(z,t)^2\right) = 0$$

Geometry		Possibility of finding a curvilineal solution	Compatibility condition
Cone and plane,	$\rho > 0$	Unlikely	Equation (16)
	$\rho = 0, \theta_o > 0$	Unlikely	Equation (12)
	$\rho = 0, \theta_o = 0$	Certain	
Parallel disk,	$\rho > 0$	Unlikely	Equations (21) and (22)
	$\rho = 0$	Certain	

if $\mathfrak{S}_1$ and $\mathfrak{S}_2$ satisfy conditions similar to Equation (21).

When inertia is neglected, or $\rho = 0$, Equation (21) is not necessary, and Equation (19c) can be integrated directly to

$$S<z\phi> = \underset{s=-\infty}{\overset{s=\infty}{\mathfrak{T}}}\left(\int_o^s r\frac{\partial \omega}{\partial z}(z, t+\tau)\,d\tau\right) = f(r,t)$$

A particular solution is then given by

$$\omega(z,t) = \frac{b-z}{b}\,\Omega\,(t)$$

where

$$\lambda(s) = \int_o^s -\frac{r}{b}\,\Omega\,(t+\tau)\,d\tau$$

For this approximation the physical components of $\underset{\sim}{S}$ depend only on r, t, so that the compatibility condition, Equation (22), is satisfied identically for all simple fluids.

DISCUSSION

Truesdell and Noll (2) summarize the presently known results pertaining to flows of the curvilineal class that exactly satisfy the equations of motion for all simple fluids. In looking over these results we can see that in every case the centripetal part of the acceleration vector and the hoop part of the stress vector depend on x^2 and point along the x^2 coordinate lines. These particular kinds of terms can be shown to be derivable from a scalar potential independent of the material properties and therefore cannot be sources of incompatibility among the equations of motion. For torsional flows, however, the centripetal acceleration and the hoop stress vectors are not so simple and generally are neither functions of x^2 alone nor directed along the x^2 coordinate lines. Therefore, these terms cannot be derived from a scalar potential unless certain integrability conditions are satisfied, and these conditions turn out to be conditions, such as Equations (12), (16), and (22), on the material properties ρ, $\mathfrak{T}$, $\mathfrak{S}_1$, $\mathfrak{S}_2$ which thereby limit the existence of exact solutions to a subclass of simple fluids.

We note that neither nonzero body forces nor edge effects have been treated in this investigation. Those body forces that can be derived from a scalar potential that does not change along a particle path can be treated in a simple way. The same is true of the edge effect if the surface $r = R_o$ is assumed to be a free surface in contact with an ideal fluid of pressure p_o; however, this boundary condition imposes a further compatibility condition on the body forces.

CONCLUSION

We can summarize the results of this paper in tabular form as follows:

NOTATION

a = constant
b = distance between parallel disks
f = function of integration
g = function of integration
g_{ij} = covariant components of the metric tensor
h_i = $\sqrt{g_{ii}}$ $i = 1, 2, 3$
p = $-1/3$ trace $\underset{\sim}{T}$, pressure
r = spherical, cylindrical coordinate
$\underset{\sim}{S}$ = $\underset{\sim}{T} + p\,\underset{\sim}{1}$, extra stress tensor
$S<ij>$ = physical components of $\underset{\sim}{S}$
s = time difference
$\underset{\sim}{T}$ = stress tensor
t = time
t' = arbitrary fixed time
$\underset{\sim}{v}$ = velocity vector
v^i = contravariant components of the velocity vector
$x^1, x^2, x^3; \bar{x}^1, \bar{x}^2, \bar{x}^3$, = orthogonal coordinates
z = cylindrical coordinate
$\mathfrak{T}$, $\mathfrak{S}_1$, $\mathfrak{S}_2$ = viscometric functionals of the material
$\underset{\sim}{1}$ = metric tensor

Greek Letters

α = angle between r and $\bar{x}^1$ coordinate lines
θ = spherical coordinate
θ_o = angle between cone and plane
$\kappa, \lambda(s)$ = quantities that determine the components of the relative finite deformation tensor for steady and unsteady curvilineal flows
$\xi\,(r, \theta, t')$ = a function whose level curves are orthogonal to those of $\omega(r, \theta, t')$
ρ = density
τ = variable of integration
ϕ = spherical, cylindrical coordinate
Ω = angular velocity
ω = $v^3 = v^\phi$ = angular velocity
τ, σ_1, σ_2 = viscometric functions of the material

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